

ON THE FUNDAMENTAL GROUP OF A COMPLETE GLOBALLY HYPERBOLIC LORENTZIAN MANIFOLD WITH A LOWER BOUND FOR THE CURVATURE TENSOR

JUN-ICHI MUKUNO

ABSTRACT. In this paper, we study the fundamental group of a certain class of globally hyperbolic Lorentzian manifolds with a positive curvature tensor. We prove that the fundamental group of lightlike geodesically complete parametrized Lorentzian products is finite under the conditions of a positive curvature tensor and the fiber compact.

1. INTRODUCTION

Calabi and Markus [4] stated the following theorem on the fundamental group of Lorentzian space forms with positive curvature:

Theorem 1.1 (Calabi–Markus [4, Theorem 1]). *Let m be an integer greater than 2. Any m -dimensional Lorentzian space form with positive curvature has a finite fundamental group.*

Wolf [13] and Kulkarni [10] generalized Theorem 1.1 for pseudo-Riemannian space forms, and Kobayashi [6, 7] studied the extension of Theorem 1.1 to reductive and solvable homogeneous spaces.

Note that Lorentzian space forms that have positive curvature are non-compact, whereas Riemannian space forms with positive curvature are compact. Theorem 1.1, however, indicates that there may be an analogy between the fundamental groups of Riemannian and Lorentzian manifolds satisfying similar geometric assumptions. In fact, Kobayashi asks whether the finiteness of the fundamental group still holds if we perturb the metric of positive constant curvature. In [8], Kobayashi proposed the following conjecture in specific terms:

Conjecture 1.1 (Kobayashi [8, Conjecture 3.8.2]). *Let m and q be positive integers with $m \geq 2q$. Assume that M is an m -dimensional geodesically complete pseudo-Riemannian manifold of index q , and that we have a positive lower bound on the sectional curvature of M . Then,*

- (1) M is never compact;
- (2) if $m \geq 3$, the fundamental group of M is always finite.

Conjecture 1.1 is analogous to Myers' theorem in Riemannian geometry, and can therefore be posed in terms of understanding the topology of pseudo-Riemannian manifolds of variable curvature. Against this aim, however, Conjecture 1.1 can

2010 *Mathematics Subject Classification.* Primary 53C50; Secondary 53C21.

Key words and phrases. fundamental group, global hyperbolicity, curvature tensor, parametrized Lorentzian product.

This work is partially supported by the Grant-in-Aid for JSPS Fellows No. 24-205.

be affirmatively solved. A proof for the case $\dim(M) = 2$ was given by Kulkarni [10, Corollary 2.10]. In the case $\dim(M) \geq 3$, Kulkarni [9] proved that the one-sided bound of the sectional curvature implies that the sectional curvature is constant. The positive constant curvature case was proved by Calabi–Markus [4] and Wolf [13], and the proof of Conjecture 1.1 is complete.

Unexpectedly, we have solved Conjecture 1.1 as a result of its curvature condition. In this paper, we reformulate Conjecture 1.1 for the Lorentzian case to cover some Lorentzian manifolds of variable curvature. Instead of the one-sided bound on the sectional curvature, Andersson and Howard [1] proposed the following curvature condition: there exists a constant k such that

$$\langle R(u, v)v, u \rangle \geq k(\langle u, u \rangle \langle v, v \rangle - \langle u, v \rangle^2)$$

for any tangent vectors u, v , where R is the curvature tensor. Following Andersson–Howard [1], we denote this condition by $R \geq k$. A subset S in a time-oriented Lorentzian manifold M is a *Cauchy hypersurface* if any inextendible timelike curve meets S at a single point. M is said to be *globally hyperbolic* if there exists a Cauchy hypersurface S in M . Bernal and Sánchez [3, Theorem 1.2] proved that a globally hyperbolic Lorentzian manifold is diffeomorphic to a product manifold of a timelike real line and some spacelike submanifold. M is said to be *non-spacelike* (resp. *lightlike*) *geodesically complete* if any inextendible, non-spacelike (resp. lightlike) geodesic is defined on the real line. We pose the following:

Conjecture 1.2. *Assume that M is a non-spacelike geodesically complete globally hyperbolic Lorentzian manifold with a compact Cauchy hypersurface of dimension $m \geq 3$. Suppose that there exists a positive constant k such that M satisfies the curvature condition $R \geq k$. Then, the fundamental group of M is finite.*

We now make some remarks on Conjecture 1.2. First, the global hyperbolicity is necessary. In fact, $(S^1 \times S^m, -g_{S^1} + g_{S^m})$ is a counterexample of Conjecture 1.2 when global hyperbolicity is excluded, where (S^m, g_{S^m}) is an m -dimensional sphere with the standard metric. Second, Beem and Ehrlich [2, Corollary 3.8] proved that the non-spacelike geodesic completeness is C^1 -stable for globally hyperbolic manifolds. Therefore, even if we perturb the metric, the assumptions of Conjecture 1.2 hold. Third, compared with Conjecture 1.1, we have omitted spacelike completeness. For the global assumption of spacelike direction, we require the compactness of Cauchy hypersurfaces. Finally, any Lorentzian space form of positive curvature satisfies the assumptions of Conjecture 1.2. This follows from the proof of Theorem 1.1.

We now give a partial solution to Conjecture 1.2. Let F be a manifold, and $\{g_t\}_{t \in \mathbb{R}}$ be a smooth family of Riemannian metrics on F . We call the Lorentzian manifold $(\mathbb{R} \times F, -dt^2 + g_t)$ a *parametrized Lorentzian product*, where t is the parameter of \mathbb{R} . We call F the *fiber*. Note that a parametrized Lorentzian product with the fiber compact is globally hyperbolic. We obtain the following theorem:

Theorem 1.2. *Let (M, g) be a lightlike geodesically complete parametrized Lorentzian product of dimension $m \geq 3$ with the fiber compact. Suppose that there exists a positive constant k such that $R \geq k$, where R is the curvature tensor of M . Then, the fundamental group $\pi_1(M)$ is finite.*

A parametrized Lorentzian product includes a kind of timelike geodesic completeness, since $\mathbb{R} \times \{p\}$ is a complete timelike geodesic for any $p \in F$. This is considered to be the reason our theorem assumes only lightlike completeness.

2. A NECESSARY CONDITION FOR THE FINITENESS OF THE FUNDAMENTAL GROUP

In this section, we consider a general setting for Theorem 1.2. We prove the following proposition:

Proposition 2.1. *Let M be a lightlike geodesically complete globally hyperbolic Lorentzian manifold. Assume that M satisfies $R \geq k$ for some positive constant k , and suppose that there exists a closed spacelike Cauchy hypersurface F in M such that the operator norm of the shape operator is bounded above by \sqrt{k} . Then, the fundamental group of M is finite.*

Let us begin the proof of Proposition 2.1. First, we calculate the sectional curvature of F . Let ∇^\top be the induced connection of the hypersurface F , and R^\top be the curvature tensor of F . Recall the following, known as the *Gauss formula*:

$$g(R(u, v)v, u) = g(R^\top(u, v)v, u) + g(S(u), u)g(S(v), v) - g(S(u), v)^2,$$

for any $u, v \in TF$. Take any 2-dimensional subspace Π in TF . Let (u, v) be an orthonormal basis of Π . We have

$$\begin{aligned} k(g(u, u)g(v, v) - g(u, v)^2) &\leq g(R^\top(u, v)v, u) + g(S(u), u)g(S(v), v) - g(S(u), v)^2 \\ (1) \qquad \qquad \qquad &\leq g(R^\top(u, v)v, u) + g(S(u), u)g(S(v), v), \end{aligned}$$

where S is the shape operator of F . The left-hand side of the inequalities is k . The requirement of the shape operator implies that $|g(S(u), u)g(S(v), v)| \leq k$. Therefore, the sectional curvature of F is non-negative.

Next, we investigate the topology of F . First, recall the following structure theorem for the fundamental group of a closed Riemannian manifold of non-negative curvature:

Theorem 2.1 (Toponogov [12], Cheeger–Gromoll [5, Theorem 3]). *Let M be a closed Riemannian manifold of non-negative sectional curvature. Then, the universal covering Riemannian manifold \tilde{M} of M can be split isometrically as $\mathbb{R}^p \times \tilde{N}$, where \tilde{N} is a closed Riemannian manifold. Moreover, the fundamental group $\pi_1(M)$ includes a free abelian subgroup \mathbb{Z}^p of finite index that acts properly discontinuously and cocompactly as a deck transformation on the Euclidean factor.*

Let g_F be the induced metric of F . We know that (F, g_F) is a closed Riemannian manifold of non-negative curvature. From Theorem 2.1, it follows that the universal covering Riemannian manifold $(\tilde{F}, g_{\tilde{F}})$ of (F, g_F) is the Riemannian product manifold of the Euclidean space and some closed Riemannian manifold \tilde{N} . Therefore, we show that the dimension of the Euclidean factor is zero. Suppose, by way of contradiction, that this dimension is not zero. Then, the fundamental group $\pi_1(F)$ has a free abelian normal subgroup \mathbb{Z}^p of finite index for some $p > 0$. Let $(\overline{F}, g_{\overline{F}})$ be the quotient Riemannian manifold $\tilde{F}/\mathbb{Z}^{p-1} = \mathbb{R} \times \overline{N}$, where \overline{N} denotes $\mathbb{R}^{p-1}/\mathbb{Z}^{p-1} \times \tilde{N}$. Then, we have the Riemannian covering map $\pi : (\overline{F}, g_{\overline{F}}) \rightarrow (F, g_F)$. Note that $g_{\overline{F}}$ is represented as the Riemannian metric $ds^2 + g_{\overline{N}}$, where s is the parameter of \mathbb{R} and $g_{\overline{N}}$ is the Riemannian metric of \overline{N} .

We now state a theorem of Bernal–Sánchez [3]:

Theorem 2.2 (Bernal–Sánchez [3, Theorem 1.2]). *Let (M, g) be a globally hyperbolic Lorentzian manifold with a spacelike Cauchy hypersurface F . Then, there exists a smooth function $\tau : M \rightarrow \mathbb{R}$ satisfying the following conditions:*

- τ is a time function, i.e., τ is strictly increasing along any future directed timelike curve;
- each level hypersurface $\tau^{-1}(t)$ is a spacelike Cauchy hypersurface for any $t \in \mathbb{R}$;
- $\tau^{-1}(0) = F$.

Let ϕ be the gradient flow $\phi : \mathbb{R} \times F \rightarrow M$ of the time function τ in the above theorem, and note that ϕ is a diffeomorphism such that $\phi(\{0\} \times F) = F$. We use the same letter g for the Lorentzian metric of $\mathbb{R} \times F$ induced from M . Then, the restricted metric $g|_{\{0\} \times F}$ is g_F . Note that the covering map $\pi : \overline{F} \rightarrow F$ extends naturally to the covering map $\text{id} \times \pi : \mathbb{R} \times \overline{F} \rightarrow \mathbb{R} \times F$. We denote the Lorentzian manifold $(\mathbb{R} \times \overline{F}, (\text{id} \times \pi)^* g)$ as $\overline{M} = (\mathbb{R} \times \overline{F}, g_{\overline{M}})$.

Let P be a submanifold of codimension 2 in M . Then, there exist two linearly independent future directed lightlike vector fields l^+, l^- perpendicular to P . P is called a *trapped surface* if $\text{div}_P(l^+) > 0$ and $\text{div}_P(l^-) > 0$. Let us recall the Penrose singularity theorem:

Theorem 2.3 (Penrose [11]). *Let M be a globally hyperbolic Lorentzian manifold of dimension $m \geq 3$ with a non-compact Cauchy hypersurface. Assume that M satisfies the following two conditions:*

- $\text{Ric}(u, u) \geq 0$ for any future directed lightlike tangent vector u ;
- P is a compact trapped surface.

Then, M is not lightlike geodesically complete.

Note that the curvature condition of Proposition 2.1 implies $\text{Ric}(u, u) \geq 0$ for any lightlike tangent vector u . Let N_0 be the submanifold $\{0\} \times \{0\} \times \overline{N}$ of $\overline{M} = (\mathbb{R} \times \mathbb{R} \times \overline{N}, g_{\overline{M}})$. We write l_+ and l_- for the normal lightlike tangent factors $n + \partial/\partial s$ and $n - \partial/\partial s$, respectively, where n is the unit normal vector of the Cauchy hypersurface $\{0\} \times \mathbb{R} \times \overline{N}$, and s is the parameter of the real line \mathbb{R} of $\{0\} \times \mathbb{R} \times \overline{N}$. Take an orthonormal basis $\{e_i\}_{i=1}^{m-1}$ of the tangent space of N_0 . Since N_0 is a totally geodesic submanifold in $\{0\} \times \mathbb{R} \times \overline{N}$, we have

$$\begin{aligned} \sum_{i=1}^{m-1} g(\nabla_{e_i} l_+, e_i) &= \sum_{i=1}^{m-1} g(\nabla_{e_i} l_-, e_i) = \sum_{i=1}^{m-1} g(\nabla_{e_i} n, e_i) \\ &= \sum_{i=1}^{m-1} g(S(e_i), e_i). \end{aligned}$$

We should remark that $g(R^\top(u, \partial/\partial s) \partial/\partial s, u) = 0$ for any unit tangent vector u of N_0 . Therefore, using inequality (1), we have $g(S(\partial/\partial s), \partial/\partial s)g(S(u), u) = k$. Without loss of generality, we can assume that $g(S(\partial/\partial s), \partial/\partial s) = \sqrt{k}$. Then, we obtain

$$\sum_{i=1}^{m-1} g(S(e_i), e_i) = (m-1)\sqrt{k} > 0.$$

Thus, N_0 is a trapped submanifold. From the Penrose singularity theorem, \overline{M} is not lightlike geodesically complete. This is a contradiction, and Proposition 2.1 has been proved.

3. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. During the proof, M is a parametrized Lorentzian manifold $(\mathbb{R} \times F, g = -dt^2 + g_t)$. For any $p \in F$, we define the curve $\gamma_p : \mathbb{R} \rightarrow M$ by $\gamma_p(t) = (t, p)$. It is easy to check that γ_p is a timelike geodesic. We denote the hypersurface $\{t\} \times F$ of M as F_t ; note that $\partial/\partial t$ is a normal vector of F_t , denoted by n . We define a second fundamental form $S : TF_t \rightarrow TF_t$ by $S(u) = \nabla_u n$, and the curvature operator $R_n : TF_t \rightarrow TF_t$ by $R_n(u) = R(u, n)n$. Then, we have the following Riccati equation:

$$(2) \quad \nabla_n(S) + S^2 + R_n = 0.$$

We obtain the following lemma:

Lemma 3.1. *For any $p \in F$ and any $u \in T_p F$ with $g_0(u, u) = 1$, let $\tilde{u}(t)$ be the parallel vector field along the timelike geodesic $\gamma_p(t)$ in M such that $\tilde{u}(0) = (0, u) \in T_0 \mathbb{R} \oplus T_p F = T_{\gamma_p(0)} \mathbb{R} \times F$. Then, we have*

$$|g(S(\tilde{u}(t)), \tilde{u}(t))| \leq \sqrt{k},$$

for any $t \in \mathbb{R}$.

Proof. The Riccati equation implies

$$\frac{\partial}{\partial t} g(S(\tilde{u}(t)), \tilde{u}(t)) = -g(R_n(\tilde{u}(t)), \tilde{u}(t)) - g(S(\tilde{u}(t)), S(\tilde{u}(t))).$$

Using the Cauchy–Schwartz inequality, we have $g(S(\tilde{u}(t)), S(\tilde{u}(t))) \geq |g(S(\tilde{u}(t)), \tilde{u}(t))|^2$. From the curvature condition, it follows that $g(R_n(\tilde{u}(t)), \tilde{u}(t)) \geq -k$. Setting $f(t) = g(S(\tilde{u}(t)), \tilde{u}(t))$, we obtain the following inequality:

$$\frac{\partial}{\partial t} f(t) \leq k - f(t)^2$$

for any $t \in \mathbb{R}$.

Let us now suppose that there exists t_0 such that $|f(t_0)| = |g(S(\tilde{u}(t_0)), \tilde{u}(t_0))| > \sqrt{k}$. Without loss of generality, we can assume that $f(t_0) > \sqrt{k}$. Then, there exists a positive number t_1 such that $f(t_0) = \sqrt{k} \coth(\sqrt{k}t_1)$. By the Riccati argument, we have

$$f(t_0 + t) \geq \sqrt{k} \coth(\sqrt{k}(t_1 + t))$$

for any $t \leq 0$. However the right-hand side of the inequality goes to infinity as t approaches $-t_1$. This is a contradiction, and the proof of Lemma 3.1 is complete.

Theorem 1.2 follows from Proposition 2.1 and Lemma 3.1.

REFERENCES

- [1] L. Andersson and R. Howard, *Comparison and rigidity theorems in semi-Riemannian geometry*, Comm. Anal. Geom. **6** (1998), no. 4, 819–877.
- [2] J. K. Beem and P. E. Ehrlich, *Geodesic completeness and stability*, Math. Proc. Cambridge Philos. Soc. **102** (1987), no. 2, 319–328.
- [3] A. N. Bernal and M. Sánchez, *Further results on the smoothability of Cauchy hypersurfaces and Cauchy time functions*, Lett. Math. Phys. **77** (2006), no. 2, 183–197.
- [4] E. Calabi and L. Markus, *Relativistic space forms*, Ann. of Math. (2) **75** (1962), 63–76.
- [5] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128.
- [6] T. Kobayashi, *Proper action on a homogeneous space of reductive type*, Math. Ann. **285** (1989), no. 2, 249–263.

- [7] ———, *On discontinuous groups acting on homogeneous spaces with noncompact isotropy subgroups*, J. Geom. Phys. **12** (1993), no. 2, 133–144.
- [8] ———, *Discontinuous groups for non-Riemannian homogeneous spaces*, Mathematics unlimited—2001 and beyond, Springer, Berlin, 2001, pp. 723–747.
- [9] R. S. Kulkarni, *The values of sectional curvature in indefinite metrics*, Comment. Math. Helv. **54** (1979), no. 1, 173–176.
- [10] ———, *Proper actions and pseudo-Riemannian space forms*, Adv. in Math. **40** (1981), no. 1, 10–51.
- [11] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57–59.
- [12] V. A. Toponogov, *Riemannian spaces containing straight lines*, Dokl. Akad. Nauk SSSR **127** (1959), 977–979.
- [13] J. A. Wolf, *The Clifford-Klein space forms of indefinite metric*, Ann. of Math. (2) **75** (1962), 77–80.

GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, CHIKUSAKU, NAGOYA 464-8602, JAPAN

E-mail address: m08043e@math.nagoya-u.ac.jp